

Module Examination

Mathematical methods, models and modelling

Time allowed: 3 hours

There are **three sections** in this examination.

In **Section 1** you should **attempt all 19 questions**. Each question is worth 2% of the total mark.

A wholly incorrectly answered question will get zero marks.

You should put all your responses on the computer-marked examination (CME) form provided, following the instructions on the next page.

In **Section 2** you should **attempt all 6 questions**. Each question is worth 5% of the total mark.

In **Section 3** you should **attempt 2 out of the 5 questions**. Each question is worth 16% of the total mark.

For **Section 1**, the assignment number for this examination is **MST210 81**. Write your rough work in the answer book(s) provided. Your rough work will not be considered by the examiners.

For **Sections 2 and 3**, write your answers in the answer book(s) provided in **pen**, though you may draw diagrams in pencil. Start your answer to each question on a new page, clearly indicating the number of the question. Crossed out work will not be marked. Write the numbers of the questions you have attempted in Section 3 on the front of your answer book(s).

Calculators are NOT permitted in this examination.
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Give yourself time to check you have completed the CME form correctly.

At the end of the examination

- Ensure Part 1 of the CME form is completed, and your personal identifier and your examination number are written on the front of each answer book used.
Failure to do so will mean that your work cannot be identified.
- Check you have selected one answer for each of questions 1 - 19 in Part 2 of the CME form.
- Place this question paper on top of the answer book(s) used. Put your signed desk record on top and fix everything together using the fastener provided.
Attach your CME form to the back of this pack using the flat paper clip provided. It is important that the flat paper clip is used with the CME form since the computer cannot read CME forms with punched holes.

Instructions for filling in the computer-marked examination (CME) form

If you do not follow these instructions, then the examiners may not be able to award you a score for Section 1. You will find one CME form provided with this paper. The invigilator has a supply of spare forms.

Writing on the CME form

- Use an HB pencil.
- To mark a cell, pencil across it, as demonstrated on the form.
- To cancel a mark, pencil in the coloured part of the cell, as demonstrated on the form.
- If you make any unwanted marks on the form that you cannot cancel clearly, then ask the invigilator for a new form, and transfer your entries to it.

Completing Part 1 of the CME form

- Write your personal identifier (NOT your examination number) and the assignment number for this examination, which is **MST210 81**, in the boxes provided.
- In the blocks headed 'PERSONAL IDENTIFIER' and 'MODULE AND ASSIGNMENT NUMBER', pencil across the cells corresponding to your personal identifier and the assignment number given above.
- On the sample CME form opposite, Part 1 has been completed for a fictitious student so that you can see how to complete this part of the form.

Completing Part 2 of the CME form

For each question (1 to 19) in Section 1:

- mark your answer by pencilling across ONE of cells A, B, C or D;
- if you think the question is unsound in any way, pencil across the 'unsound' cell (U), *as well as* pencilling across your chosen answer cell;
- ignore the 'don't know' cell (?), as it is always best to mark an answer.

PART 1

PERSONAL IDENTIFIER
(NOT Examination Number)

A7654321

ASSIGNMENT NUMBER
(as given on the question paper)

MST210 81

NAME and INITIALS

A. STUDENT

IMPORTANT NOTES

1. Please check that all of this form has been completed including the coloured boxes in Part 1 with your Personal Identifier, Module and Assignment Number.
2. Only use an HB pencil to complete this form and press firmly.
3. Shown opposite are examples of the correct marks to use. Do not use any other types of mark.

**PLEASE DO NOT PUNCH HOLES
IN THIS FORM**

Fold this form INWARDS along the dotted line

PERSONAL IDENTIFIER

A	0	0	0	0	0	0	0
B	1	1	1	1	1	1	1
C	2	2	2	2	2	2	2
D	3	3	3	3	3	3	3
E	4	4	4	4	4	4	4
F	5	5	5	5	5	5	5
G	6	6	6	6	6	6	6
H	7	7	7	7	7	7	7
K	8	8	8	8	8	8	8
L	9	9	9	9	9	9	9
M							X
N							
P							
R							
S							
T							
U							
V							
W							
X							
Y							

CORRECT MARK

Pencil across
between dots



CANCELLATION MARK

To cancel a mark
pencil in the
coloured part of
the cell. Do not rub out



MODULE AND ASSIGNMENT NUMBER

A	A	A	A	0	0	0	0
B	B	B	B	1	1	1	1
C	C	C	C	2	2	2	2
D	D	D	D	3	3	3	3
E	E	E	E	4	4	4	4
F	F	F	F	5	5	5	5
G	G	G	G	6	6	6	6
H	H	H	H	7	7	7	7
K	K	K	K	8	8	8	8
L	L	L	L	9	9	9	9
M	M	M	M				
N	N	N	N				
P	P	P	P				
R	R	R	R				
S	S	S	S				
T	T	T	T				
U	U	U	U				
W	W	W	W				
X	X	X	X				
Y	Y	Y	Y				
Z	Z	Z	Z				

OFFICE USE ONLY

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PART 2

ANSWER

ANSWER

ANSWER

1	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
2	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
3	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
4	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
5	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
6	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
7	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
8	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
9	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
10	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
11	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
12	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
13	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
15	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
16	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
17	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
18	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
19	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
20	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
21	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
22	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
23	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
24	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
25	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
26	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
27	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
28	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
29	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
30	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
31	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
32	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
33	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
34	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
35	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
36	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
37	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
38	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
39	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
40	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
41	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
42	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
43	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
44	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
45	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
46	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
47	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
48	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
49	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
50	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
51	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
52	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
53	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
54	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
55	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
56	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
57	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
58	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
59	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
60	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z

Section 1

You should **attempt all questions** marking your answers in **pencil** on the CME form. **Each question is worth 2%.**

Question 1

Select the option that gives the general solution (involving the arbitrary constants A and B) of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0.$$

Options

- A** $y = Ae^{-3x} + Be^x$ **B** $y = e^{-x}(A \cos(2x) + B \sin(2x))$
C $y = A \cos(2x) + B \sin(2x)$ **D** $y = e^{-2x}(A \cos(x) + B \sin(x))$

Question 2

Select the option that gives the angle between the vectors $\mathbf{i}$ and $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

Options

- A** 0 **B** $\arccos\left(\frac{1}{\sqrt{3}}\right)$ **C** $\arccos\left(\frac{1}{3}\right)$ **D** $\frac{\pi}{2}$

Question 3

The position of a car on a fairground ride at time t is given by

$$\mathbf{r}(t) = \cos(2\pi t)\mathbf{i} + \sin(2\pi t)\mathbf{j} + t\mathbf{k}.$$

Select the option that gives the magnitude of the acceleration of the car.

Options

- A** 2 **B** 2π **C** $4\pi^2$ **D** $16\pi^4$

Question 4

Consider the following augmented matrix with two rows labelled $\mathbf{R}_1$ and $\mathbf{R}_2$.

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ -2 & 2 & 1 \end{array} \right) \begin{array}{l} \mathbf{R}_1 \\ \mathbf{R}_2 \end{array}$$

Select the option that gives the first step of the Gaussian elimination method applied to this augmented matrix.

Options

- A** $\mathbf{R}_{2a} = \mathbf{R}_2 - 2\mathbf{R}_1$ **B** $\mathbf{R}_{2a} = \mathbf{R}_2 + 2\mathbf{R}_1$
C $\mathbf{R}_{2a} = \mathbf{R}_2 + \mathbf{R}_1$ **D** $\mathbf{R}_{2a} = \mathbf{R}_2 - \frac{1}{3}\mathbf{R}_1$

Question 5

The eigenvalues of a 3×3 matrix $\mathbf{A}$ are 1, 3 and 5. Select the option that gives the eigenvalues of $(\mathbf{A} - 2\mathbf{I})^{-1}$.

Options

- A** 1, $\frac{1}{3}$, $\frac{1}{5}$ **B** -1, 1, 3 **C** -1, 1, $\frac{1}{3}$ **D** $\frac{1}{2}$, $2\frac{1}{2}$, $4\frac{1}{2}$

Question 6

The system of differential equations

$$\dot{x} = 17x - 9y,$$

$$\dot{y} = -9x - 7y,$$

has general solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{20t} + \beta \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-10t}.$$

Select the option that gives the particular solution of this system that satisfies $x = 1$ and $y = -7$ when $t = 0$.

Options

- A** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{20t} + \begin{pmatrix} -2 \\ -6 \end{pmatrix} e^{-10t}$
- B** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \end{pmatrix} e^{20t} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-10t}$
- C** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{20t} + \begin{pmatrix} -2 \\ -6 \end{pmatrix} e^{-10t}$
- D** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \end{pmatrix} e^{20t} + \begin{pmatrix} -1 \\ -3 \end{pmatrix} e^{-10t}$

Question 7

The function f is defined by

$$f(u, v) = \ln(u^2 + v^2) - uv.$$

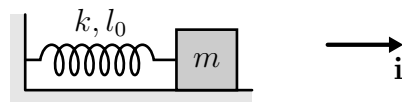
Select the option that gives $\partial f / \partial u$.

Options

- A** $\frac{1}{u^2 + v^2} - u$ **B** $\frac{2u}{u^2 + v^2} - u$ **C** $\frac{2u}{u^2 + v^2} - v$ **D** $\frac{1}{u^2 + v^2} - v$

Question 8

A particle is attached to a spring of natural length l_0 and stiffness k , and lies on a smooth horizontal table. The other end of the spring is attached to a fixed point, and the spring is compressed by $\frac{3}{4}l_0$. The situation is illustrated in the figure below, which also shows the unit vector $\mathbf{i}$.



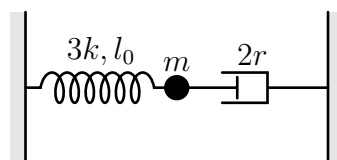
Select the option that gives the horizontal force acting on the particle while it is in this position.

Options

- A $\frac{3}{4}kl_0 \mathbf{i}$ B $-\frac{3}{4}kl_0 \mathbf{i}$ C $\frac{1}{4}kl_0 \mathbf{i}$ D $-\frac{1}{4}kl_0 \mathbf{i}$

Question 9

A particle of mass m is attached to a model spring of natural length l_0 and stiffness $3k$, and to a model damper with damping constant $2r$ ($r > 0$), as shown below.



Select the option that gives the condition that means that the system will oscillate if displaced from equilibrium.

Options

- A $r^2 < mk$ B $r^2 > mk$ C $r^2 < 3mk$ D $r^2 > 3mk$

Question 10

The equation of motion of a system of particles and springs is

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Select the option that gives the normal mode angular frequencies of the system.

Options

- A 1 and 2 B 1 and 4 C 1 and 8 D 1 and 16

Question 11

The following system of linear differential equations has an equilibrium point at $(0, 0)$:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Select the option that describes the nature of this equilibrium point.

Options

A Sink **B** Spiral source **C** Spiral sink **D** Saddle

Question 12

Select the option that could be the Fourier series of the function $f(t)$ defined by

$$f(t) = t, \quad -\pi \leq t < \pi,$$

$$f(t + 2\pi) = f(t).$$

Options

A $F(t) = -2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nt)$ **B** $F(t) = 1 - 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nt)$

C $F(t) = -2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cos(nt)$ **D** $F(t) = 1 - 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cos(nt)$

Question 13

Consider the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},$$

with boundary conditions $u(0, t) = 0$ and $u_x(2\pi, t) = 0$ for $t \geq 0$.

Applying the method of separation of variables with $u(x, t) = X(x)T(t)$ gives the ordinary differential equations

$$X'' = \mu X, \quad \dot{T} = \mu T,$$

where μ is a non-zero separation constant.

Select the option that gives the boundary conditions that must hold to give non-trivial solutions of these ordinary differential equations.

Options

A $X(0) = 0, \quad X'(2\pi) = 0$ **B** $X'(0) = 0, \quad X(2\pi) = 0$

C $T(0) = 0, \quad \dot{T}(2\pi) = 0$ **D** $\dot{T}(0) = 0, \quad T(2\pi) = 0$

Question 14

A scalar field is defined in spherical coordinates by

$$g(r, \theta, \phi) = r \sin \phi \sin \theta.$$

Select the option that gives the $\mathbf{e}_\theta$ -component of ∇g .

Options

- A** $\sin \phi \cos \theta$ **B** $r \sin \phi \cos \theta$ **C** $r \sin \theta \cos \phi$ **D** $\sin \theta \cos \phi$

Question 15

Select the option that gives $\nabla \cdot \mathbf{F}$ where $\mathbf{F}$ is defined in spherical coordinates by

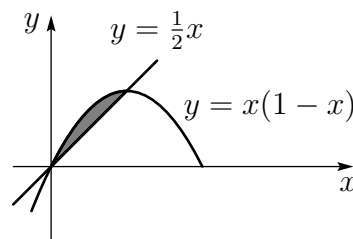
$$\mathbf{F}(r, \theta, \phi) = r^3 \sin \theta \mathbf{e}_r.$$

Options

- A** $3r^2$ **B** $5r^2$ **C** $3r^2 \sin \theta$ **D** $5r^2 \sin \theta$

Question 16

Select the option that gives an integral that can be used to calculate the shaded area in the figure below, which depicts the region above the line $y = \frac{1}{2}x$ and below the curve $y = x(1 - x)$.



Options

- A** $\int_{x=0}^{x=1/2} \left(\int_{y=x/2}^{y=x(1-x)} 1 \, dy \right) dx$ **B** $\int_{x=0}^{x=1} \left(\int_{y=x/2}^{y=x(1-x)} 1 \, dy \right) dx$
- C** $\int_{x=0}^{x=1/2} \left(\int_{y=0}^{y=x(1-x)} 1 \, dy \right) dx$ **D** $\int_{x=0}^{x=1} \left(\int_{y=0}^{y=x(1-x)} 1 \, dy \right) dx$

Question 17

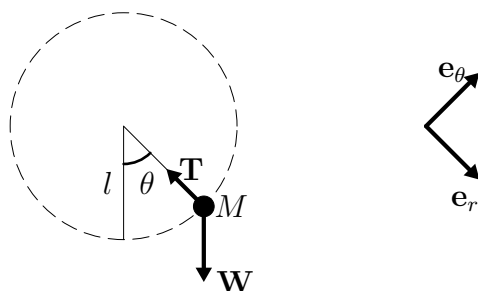
Two particles of mass m moving with speed u collide and coalesce into a particle that moves with speed $u/\sqrt{2}$. Select the option that describes this collision.

Options

- A** Elastic with coefficient of restitution 0
- B** Inelastic with coefficient of restitution 0
- C** Elastic with coefficient of restitution 1
- D** Inelastic with coefficient of restitution 1

Question 18

A particle of mass M is connected to a model string of length l and is moving in a vertical circle, as illustrated below. Tangential and radial unit vectors are also shown. There are no frictional forces.



Select the option that gives the radial component of the equation of motion of the particle when it makes an angle θ with the downward vertical, as shown. (The magnitude of the tension in the string at this instant is $|\mathbf{T}|$.)

Options

A $Ml\dot{\theta}^2 = Mg \cos \theta - |\mathbf{T}|$

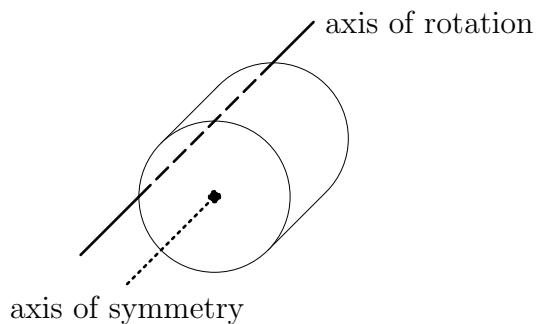
B $Ml\dot{\theta}^2 = |\mathbf{T}| - Mg \cos \theta$

C $l\ddot{\theta} = g \sin \theta$

D $l\ddot{\theta} = -g \sin \theta$

Question 19

A solid cylinder of mass M and radius R rotates about an axis that is parallel to the centre line of the cylinder and is on the surface of the cylinder, as shown in the diagram below. The solid line (dashed where it is hidden by the cylinder) shows the axis of rotation, and the centre line of the cylinder (where visible) is indicated by a dotted line.



Select the option that gives the moment of inertia of the cylinder about the axis of rotation.

Options

A $\frac{1}{2}MR^2$

B MR^2

C $\frac{3}{2}MR^2$

D $2MR^2$

Section 2

You should **attempt all questions**, write in **pen** and start your answer to each question on a new page. **Include all your working**, as some marks are awarded for this. **Each question is worth 5%.**

Question 20

Solve the initial-value problem

$$\frac{dy}{dx} = (y + 1) \sin x, \quad y(0) = 0,$$

expressing your answer in the form $y = f(x)$. [5]

Question 21

Consider the function

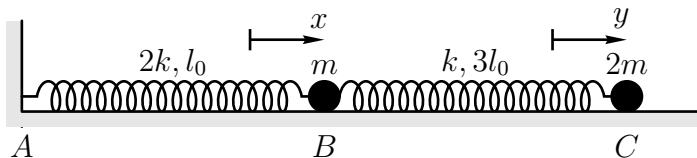
$$f(x, y) = 2x^2 - 4xy - 3y^2 - 4x + 4y.$$

(a) Find the stationary point of f . [3]

(b) Determine the nature of this stationary point. [2]

Question 22

Two model springs, AB and BC , have natural lengths l_0 and $3l_0$, and stiffnesses $2k$ and k , respectively. A particle of mass m is attached to the springs at B , and a particle of mass $2m$ is attached at C . The point A is fixed to a rigid support, and the particles are constrained to move along the line through A and C on a frictionless horizontal plane. Let x and y be the displacements of particles B and C from their respective equilibrium positions. This situation is shown in the figure below.



Find the differential equations satisfied by x and y , expressing your answer in matrix form. [5]

Question 23

The function f is defined for $0 \leq t < 1$ by the equation

$$f(t) = \frac{1}{2} - t.$$

Sketch graphs of the even and odd extensions of f that have period 2, and hence write down their definitions. [5]

Question 24

Calculate the volume of the cylindrically symmetric solid that is formed by the portion of the surface $z = 4 - x^2 - y^2$ that lies above the (x, y) -plane. [5]

Question 25

A particle of mass m moving with velocity $u\mathbf{i}$ collides with a particle of mass m moving with velocity $u\mathbf{j}$. After the collision, the two particles coalesce and move with velocity $v\mathbf{i} + w\mathbf{j}$.

- (a) Use conservation of momentum to calculate the speed of the particle after the collision. [3]
- (b) Calculate the ratio of the kinetic energy after collision to the kinetic energy before the collision. [2]

Section 3

You should attempt **two questions**. If you attempt more, the scores from your best two questions will count. Write in **pen** and start your answer to each question on a new page. **Include all your working**, as some marks are awarded for this. **Each question is worth 16%.**

Question 26

Find the solution of the simultaneous linear differential equations

$$\frac{dx}{dt} = -4x + 6y - 3e^{-t},$$

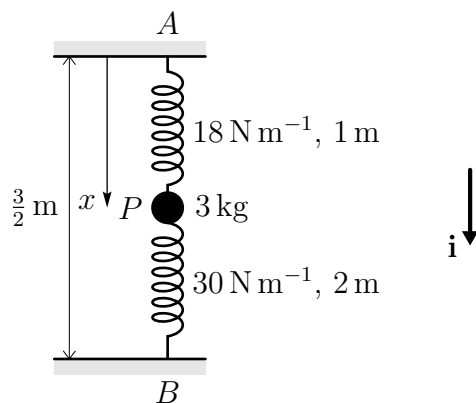
$$\frac{dy}{dt} = -9x + 11y + 3e^{-t},$$

that satisfies the initial conditions $x(0) = 0$, $y(0) = 0$.

[16]

Question 27

A particle P of mass 3 kg is constrained to move in a vertical straight line between the fixed points A and B , with A at a distance $\frac{3}{2}$ m vertically above B . The particle is attached to two springs, AP of natural length 1 m and stiffness 18 N m^{-1} , and PB of natural length 2 m and stiffness 30 N m^{-1} . Assume that air resistance is negligible. The unit vector $\mathbf{i}$ is defined to point vertically downwards, as shown in the diagram below. At time t , the displacement of P from A is $x(t)$.



- Draw a force diagram showing all the forces acting on the particle. Define each force that you introduce. [3]
- At time t , express each force in terms of the given unit vectors. [4]
- Find the value of x in terms of g , the magnitude of the acceleration due to gravity, when the system is in equilibrium. [2]
- Determine a differential equation of vertical motion when the particle is in motion, simplifying your result as far as possible. [2]
- Solve the equation of motion when the particle is released from rest at a distance of $\frac{1}{4}$ m above the equilibrium position found in part (c). [3]
- Sketch the graph of x against t for the range $0 \leq t \leq \pi$, showing all the salient features of the motion. [2]

Question 28

Consider the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha} \frac{\partial u}{\partial t},$$

where α is a positive constant, subject to the boundary conditions

$$u(0, t) = u(\pi, t) = 0, \quad t \geq 0.$$

- (a) Show, by separating the variables, that any function $u(x, t)$ that satisfies the partial differential equation and the boundary conditions may be written in the form

$$u(x, t) = \sum_{n=1}^{\infty} B_n e^{-\alpha n^2 t} \sin(nx),$$

where the B_n are constants. You may assume that the constant introduced by using the separation of variables method is negative. [13]

- (b) Find an integral expression for the general coefficient B_n so that $u(x, t)$ also satisfies the initial condition

$$u(x, 0) = f(x), \quad 0 < x < \pi,$$

where $f(x)$ is a given function. [3]

Question 29

Consider the vector fields defined throughout the whole of space by

$$\mathbf{F} = (xz^2 + y)\mathbf{i} + (-yz^2 + x)\mathbf{j} + z(x^2 - y^2)\mathbf{k},$$

$$\mathbf{G} = (xz^2 - y)\mathbf{i} + (yz^2 + x)\mathbf{j} + z(x^2 - y^2)\mathbf{k}.$$

- (a) Use the curl test to show that $\mathbf{F}$ is conservative. [5]
(b) Calculate the value of the line integral of $\mathbf{G}$ around the circular path

$$x = \cos t, \quad y = \sin t, \quad z = 0 \quad (0 \leq t \leq 2\pi),$$

from the point $(1, 0, 0)$ back to the same point. Deduce that $\mathbf{G}$ cannot be the gradient of a scalar field, briefly justifying your reasoning. [5]

- (c) Calculate the scalar line integral of $\mathbf{F}$ along the line segment

$$x = at, \quad y = bt, \quad z = ct \quad (0 \leq t \leq 1),$$

where (a, b, c) is any given point, and use this result to find a scalar field $U(x, y, z)$ such that

$$\mathbf{F} = -\nabla U \quad \text{and} \quad U(0, 0, 0) = 0. \quad [6]$$

Question 30

Consider a pendulum, attached to a fixed point O , whose rod is of length l and whose bob is of mass m . In addition to gravity, the pendulum is subject to a linear resistive force of magnitude kv , where k is a constant and v is the speed of the pendulum bob. Use a polar coordinate system with origin at O , and let θ be the angle measured anticlockwise between the pendulum rod and the downward vertical. Consider the motion of the pendulum when θ is positive and increasing.

- (a) Draw a picture of the system showing the choice of axes, and draw a force diagram showing all of the forces acting on the pendulum bob. [2]
- (b) Write all the forces and the position of the pendulum bob relative to O in terms of scalars and the polar unit vectors. [3]
- (c) Calculate the total torque about O in terms of $m, g, k, l, \theta, \dot{\theta}$ and the polar unit vectors. [5]
- (d) Calculate the angular momentum of the pendulum bob about O . [3]
- (e) Hence show that the equation of motion of the pendulum bob is given by

$$\ddot{\theta} + \frac{k}{m}\dot{\theta} + \frac{g}{l}\sin\theta = 0. \quad [3]$$

[END OF QUESTION PAPER]